χ = the angle between **r** and **ω**
ϵ = the angle between **ω** and **A** (a fixed quantity for most satellites)
ζ = azimuthal angle of spin of **A** about **ω**
κ = the angle of rotation of **ω** about **r**, which gives the orientation of **ω** with respect to the **s**, **r** plane (defined similar to φ)

In Fig. 2, these quantities are shown, and a coordinate system (X,Y,Z) is defined, from which φ and κ are defined. As **A** rotates about **ω**, it coincides with the **ω**, **r** plane on two occasions. The zero value of ζ is taken to be the situation for which **A** lies in the **ω**,**r** plane, and its motion is in the positive κ direction. Then,

$$\cos\eta = \frac{(r - \cos\vartheta)}{(r^2 + 1 - 2r\cos\vartheta)^{1/2}}(\cos\chi\cos\epsilon - \sin\chi\sin\epsilon\cos\zeta) + \frac{\sin\vartheta}{(r^2 + 1 - 2r\cos\vartheta)^{1/2}}[\cos\varphi\cos\kappa\cos\epsilon\sin\chi + \sin\varphi\sin\kappa\cos\epsilon\sin\chi + \cos\varphi\cos\kappa\sin\epsilon\cos\chi\cos\zeta + \sin\varphi\sin\kappa\sin\epsilon\cos\chi\cos\zeta + \sin\varphi\cos\kappa\sin\epsilon\sin\zeta - \cos\varphi\sin\kappa\sin\epsilon\sin\zeta] \quad (4)$$

Setting the right side of Eq. (4) to zero, one has

$$\cos\varphi_{m2} = -FG \pm H(G^2 + H^2 - F^2)^{1/2}/(G^2 + H^2) \quad (5)$$

where

$$F = [(r - \cos\vartheta)/\sin\vartheta](\cos\chi\cos\epsilon - \sin\chi\sin\epsilon\cos\zeta)$$
$$G = (\cos\kappa\cos\epsilon\sin\chi + \cos\kappa\sin\epsilon\cos\chi\cos\zeta - \sin\kappa\sin\epsilon\sin\zeta)$$
$$H = (\sin\kappa\cos\epsilon\sin\chi + \sin\kappa\sin\epsilon\cos\chi\cos\zeta + \cos\kappa\sin\epsilon\sin\zeta)$$

Since φ_{m1} is determined by the source function that is symmetric about the **s**,**r** plane, the values of φ in the range $0 \leqslant \varphi \leqslant 2\pi$ which contribute to the input are $0 \leqslant \varphi \leqslant \varphi_{m1}$, $(2\pi - \varphi_{m1}) \leqslant \varphi \leqslant 2\pi$, and $\varphi_{m1} \leqslant \pi$. However, the solution of Eq. (5) for φ_{m2} yields two roots symmetric about the **ω**,r plane. Since the function is multivalued, the problem is to find them. For any given problem these can be determined upon careful examination of the physical picture and with the aid of the following: 1) one value φ_{m2}', say, must lie in the range $\kappa \leqslant \varphi \leqslant \pi + \kappa$, and the other, φ_{m2}'', must lie in the range $\pi + \kappa \leqslant \varphi \leqslant 2\pi + \kappa$; and 2) the values of φ which contribute to the input are $\kappa \leqslant \varphi \leqslant \varphi_{m2}'$ and $(\kappa + 2\pi - \varphi_{m2}') \leqslant \varphi \leqslant 2\pi + \kappa$. In practice, the computer program is written in such a way that it is not necessary actually to determine the limits. The only values of φ which contribute are those lying in the overlapping regions of the two ranges given by φ_{m1} and φ_{m2}. Therefore, for the φ integration, the program that has been used is written in such a way that, for each value of φ (remember that the computer calculates the function for incremental steps of φ and adds them), the computer makes a check of both $\cos\beta$ and $\cos\eta$. If *both* are $\leqslant 1$ and >0, then that value of φ contributes, and the complete computation is made and stored. If one or both is $\leqslant 0$, a zero is entered. In this manner the computer runs through the entire range of φ from 0 to 2π.

For the special case when $\chi = 0$ (i.e., the spin axis coincides with **r**), the expression to be used for $\cos\eta$ is

$$\cos\eta = \frac{\sin\epsilon\sin\vartheta(\sin\zeta\sin\varphi + \cos\zeta\cos\varphi)}{(r^2 + 1 - 2r\cos\vartheta)^{1/2}} + \frac{(r - \cos\vartheta)\cos\epsilon}{(r^2 + 1 - 2r\cos\vartheta)^{1/2}} \quad (6)$$

Equation (6) follows from Eq. (4) by making the substitutions $\chi = 0$ and $\kappa = 0$. The latter is required because, for the case when $\chi = 0$, the angle κ has no meaning and therefore is given here the value of zero.[3] In addition, it might be well to point out here that many combinations of values of the parameters ϵ, χ, κ yield an identical physical picture as another set. Therefore, the only range of values for the parameters that need be considered is $0 \leqslant \chi \leqslant \pi$, $0 \leqslant \epsilon \leqslant \pi/2$, and $0 \leqslant \kappa \leqslant \pi$. For any value of $\epsilon > \pi/2$, the same physical picture is obtained if χ is replaced by $(\pi - \chi)$, ϵ by $(\pi - \epsilon)$, and κ by $(\pi + \kappa)$. For any value of $\kappa > \pi$, one can replace it by $(2\pi - \kappa)$, all other parameters remaining the same. The fact that a value of $\chi > \pi$ is equivalent to $(2\pi - \chi)$ is obvious.

The average over a spin period $\langle P \rangle$ defined by

$$\langle P \rangle = \frac{1}{2\pi}\int_0^{2\pi} P(\zeta)d\zeta$$

is done quite simply by the computer.

Discussion

In the foregoing, the equation for P, even though it has not been solved explicitly, is an exact expression only for the geometric aspects of the problem. The assumption that earth is a spherical, diffuse reflector is necessary if the equation is not to be much more complicated than it is. However, these assumptions would seem to be not as serious as the approximations that the reflectivity is latitude- and longitude-independent and that there is no time-dependency. This undoubtedly is not true. If the ϑ and φ dependence were known, the expression could be modified readily, resulting in essentially no additional labor for the computer integrations. The time variation is, of course, much greater, depending upon such things as cloud cover, cloud location with respect to the sun, etc. However, in assuming a certain amount of spatial uniformity (as is done in the present model), time changes can be handled because the average albedo enters the expressions only as a multiplicative constant, and the result can be changed accordingly.

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Real Gas Performance of Helium Drivers

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For lunar and planetary entry problems, a basic research tool in problems involving convective heat transfer and radiative heat transfer is the electrically driven hypervelocity shock tube.[1, 2] The driver contains helium, which is heated by the rapid discharge of electrical energy. Previous analysis of the performance of electrically driven shock tubes has been limited to the perfect gas assumption because of the lack of thermodynamic data on helium. During the course of the

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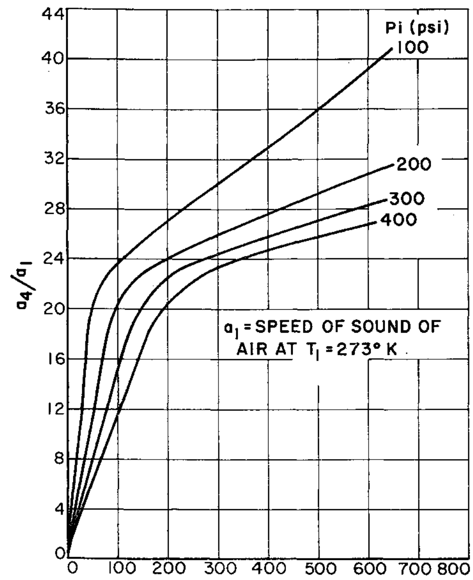

Fig. 1 Speed of sound ratio for energy input to a constant volume of helium

development of a hypervelocity research facility at Jet Propulsion Laboratory, it was felt that more accurate calculations were needed. The recent publication at Harvard[3] of a Mollier diagram for helium has made it possible to calculate the real gas properties of the helium driver as a function of energy input. Some extrapolation of the Harvard results was needed in order to complete the computation.

These calculations should make it easier to predict more accurately the performance of hypervelocity shock tubes. Figures 1–3 give the final temperature and pressure and the sound speed ratio to air of the driver as a function of the initial conditions and the energy input per unit volume. The initial energy of the gas has been neglected.

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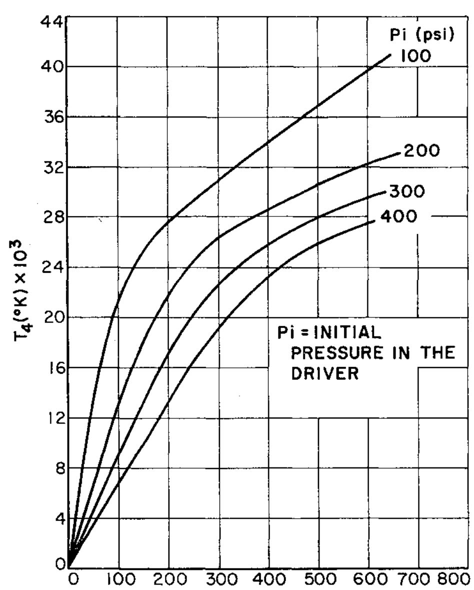

Fig. 2 Resultant temperature for an energy input to a constant volume of helium

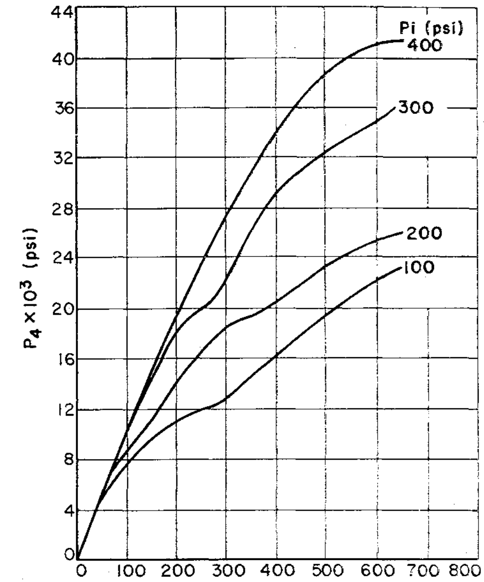

Fig. 3 Resultant pressure for an energy input to a constant volume of helium

Discrete Element Approach to Structural Instability Analysis

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Reference 1 suggests a matrix displacement formulation of the beam-column problem involving an assumption of linear displacements between element end points in the calculation of the axial load effects. The present note derives a general procedure for including instability effects in element force-displacement relationships applicable to both beams and plates and demonstrates that it can be used to obtain results to a high degree of accuracy with relatively few node points.

In the conventional approach to matrix displacement analysis, structural systems are idealized as assemblages of discrete elements. Relationships between the element juncture point forces $\{F\}$ and displacements $\{\delta\}$ are stated in the form $\{F\} = [k]\{\delta\}$, where $[k]$ is the element stiffness matrix. Upon evaluation, the element relationships are combined, in accordance with the requirements of node point equilibrium and compatible displacements, to yield

$$\{P\} = [K]\{\Delta\} \tag{1}$$

where $\{P\}$ are the node point external loads, $[K]$ is the stiffness matrix of the assembled idealization, and $\{\Delta\}$ are the node point displacements. The inverse of $[K]$ (after it has been modified in recognition of the geometric boundary conditions) is the set of structural influence coefficients. This note treats cases where Eq. (1) relates only the flexural forces and displacements, i.e., the midplane stress system is deter-

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